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INTEGRAL EQUATION FOR STRESS CONCENTRATION AT THE EDGE OF A PLANE CRACK OF ARBITRARY CONTOUR

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Equations are derived for stress concentration near a crack of closed contour lying in a plane. A system of one-dimensional integral equations for the concentration factor is obtained. The right sides of the equations contain the initial approximation — a solution of the problem of a circular crack whose sides are acted upon by nonaxisymmetric loading.

In solving mixed problems for harmonic functions, it is necessary to evaluate functions on boundary segments on which their values are not specified in boundary-value problems. For example, in the problem of stationary filtration of a liquid by the Darcy rule into the depth of a homogeneous porous half-space through a permeable spot on the surface, the pressure of the overlying liquid on the spot is known, and in the impermeable part of the boundary outside the permeable spot, the normal component of the vector velocity is equal to zero. A calculation of the liquid velocity normal to the permeable part of the boundary is required to determine the liquid flow rate. In the mixed problem of a brittle opening-mode crack, displacements on the crack extension in the crack plane and varying stress normal to the crack are specified. An interesting quantity in this problem is the stress on the crack extension since from the stress-intensity factor, it is possible to determine the stable form of the crack.

Usually, the boundary equations of potential theory are employed in determining such quantities. In the crack problem, two methods of calculation are possible. The first method uses the Fredholm equation of the first kind, in which the displacement w on the crack is expressed in terms of the stress σ as

$$w(r,\vartheta,0) = \frac{1}{2\pi A}\int\limits_0^{2\pi}\int\limits_0^{\infty}\rho\sigma(\rho,\alpha,0)\,\frac{d\rho\,d\alpha}{R(\rho,r,\vartheta-\alpha)}. \tag{1}$$

Here and below, $z=0$, $r<l(\vartheta)$ is the position of the crack in the cylindrical coordinates (r,ϑ,z), $R(\rho,r,\vartheta-\alpha)$ is the distance between the points (r,ϑ) and (ρ,α), and $A=\mu/(1-\nu)$, where μ and ν are the shear modulus and Poisson's constant.

Since, by virtue of symmetry, the displacement is equal to zero on the crack extension $z=0$ and $r>l(\vartheta)$, we obtain an equation for the unknown function $\sigma=\sigma_+$ at $z=0$ and $r>l(\vartheta)$ [the values of $\sigma=\sigma_-$ for $r<l(\vartheta)$ are specified]. The instability of calculation schemes for equations of this type and the unboundedness of the region in which solutions are sought hinder the search for singular solutions, and the sought function is a singular solution.

In the second method, an equation for the boundary solution is obtained from the equation given above by inversion if the integrals are understood as an integral transformation of the function σ to w:

$$\sigma(r,\vartheta,0) = \frac{A}{2\pi}\lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}\int\limits_0^{2\pi}\int\limits_0^{l(\alpha)}\rho w(\rho,\alpha,0)\,\frac{\varepsilon\,d\rho\,d\alpha}{[\varepsilon^2+R^2(\rho,r,\vartheta-\alpha)]^{3/2}}. \tag{2}$$

The parameter ε is introduced to reduce the singularity of the kernel.

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For $r < l(\vartheta)$, this expression is an integral equation for displacement. (It contains the derivative of the double-layer potential with respect to the normal.) It is stable in calculations, but the high degree of singularity leads to difficulties in numerical implementation. In addition, the stress is evaluated from the displacement found, and this introduces an additional error in calculations of the stress-intensity factor.

In the present paper, to calculate the stress-intensity factor, we derive a modified integral boundary equation that is "intermediate" between Eqs. (1) and (2).

By means of Papkovich representations, determination of the opening-mode crack parameters reduces to seeking one harmonic function $f(r, \vartheta, z)$. The displacements and stresses can be expressed in terms of this function. For example, for the normal components of the displacement and stress vectors on a site with normal parallel to the z axis, we have

$$w = 2\Big(1 - \nu - \frac{z}{2}\frac{\partial}{\partial z}\Big)\frac{\partial f}{\partial z}, \qquad \sigma = 2\mu\Big(1 - z\frac{\partial}{\partial z}\Big)\frac{\partial^2 f}{\partial z^2}.$$

In the problem of an opening-mode crack whose points $r < l(\vartheta)$ lie in the plane $z = 0$, the function f should satisfy the conditions

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{2\mu}\sigma_-(r, \vartheta) \quad [r < l(\vartheta)], \qquad \frac{\partial f}{\partial z} = 0 \quad [r > l(\vartheta)].$$

It is assumed that the crack contour does not have angular points, and, for simplicity, it is considered star-shaped.

We examine the upper half-space. Expanding the functions f, w, and σ in complex Fourier series in the angular coordinate and performing a Hankel transformation with kernel $rJ_n(qr)$ (q is a transformation parameter) for the factors w_n and σ_n, for each harmonics we obtain solutions that decrease exponentially along z. In the crack plane, the Hankel images of the Fourier coefficients are related by the condition (for $z = 0$, the dependence on this argument for all functions is omitted below)

$$\sigma_n^H(q) = -Aqw_n^H(q).$$

In this equality, we convert to preimages. We invert the Hankel images with integer indices using the inversion formula for the Hankel transformation with half-integer indices. Multiplying the last equality by $\sqrt{q}J_{n+1/2}(qx)$ and integrating it with respect to q taking into account the formula for discontinuous integrals (see [1, formula 6.575.1]), we have

$$\int_0^\infty q^{\mu-\nu} J_{\nu+1}(aq)J_\mu(bq)\,dq = \begin{cases} 0, & a < b, \\[2mm] \dfrac{(a^2 - b^2)^{\nu-\mu} b^\mu}{2^{\nu-\mu}a^{\nu+1}\Gamma(\nu - \mu + 1)}, & a > b, \end{cases}$$

where Γ is a gamma function.

We obtain the following equations for the coefficients of the harmonics:

$$\int_0^x \frac{\rho\sigma_n(\rho)}{\sqrt{x^2 - \rho^2}}\Big(\frac{\rho}{x}\Big)^n d\rho = Ax\int_x^\infty \frac{\partial}{\partial\rho}\Big[w_n(\rho)\Big(\frac{x}{\rho}\Big)^n\Big]\frac{d\rho}{\sqrt{\rho^2 - x^2}}. \tag{3}$$

After summation over n in infinite limits with weight $\exp(in\vartheta)$ taking into account the expressions of the Fourier coefficients in terms of the functions expanded in a series, the equations reduce to the following integral equation with respect to the unknown half-opening of the crack w and the stress on the crack extension $\sigma = \sigma_+$ ($\sigma = \sigma_- + \sigma_+$, where σ_- is specified):

$$\begin{gathered}\int_0^{2\pi}\int_0^x \frac{\rho\sigma(\rho,\alpha)\sqrt{x^2 - \rho^2}\,d\rho\,d\alpha}{R^2(\rho, x, \vartheta - \alpha)} = Ax\int_0^{2\pi}\int_x^{l(\alpha)} \frac{\partial}{\partial\rho}\Big[\frac{w(\rho,\alpha)(\rho^2 - x^2)}{R^2(\rho, x, \vartheta - \alpha)}\Big]\frac{d\rho\,d\alpha}{\sqrt{\rho^2 - x^2}}, \\ R^2(\rho, x, \vartheta - \alpha) = \rho^2 - 2\rho x\cos(\vartheta - \alpha) + x^2.\end{gathered} \tag{4}$$

In the particular case of a circular opening-mode crack $l(\vartheta) = L = \text{const}$ that opens under the action of nonaxisymmetric loading on the sides, the solution of this equation is known (see, for example, [2]):

$$w(r,\vartheta) = -\frac{2}{\pi A}\int_r^L \frac{1}{\sqrt{x^2-r^2}}\int_0^x \frac{\rho Q(\rho, x^2/r, \vartheta)\, d\rho\, dx}{\sqrt{x^2-\rho^2}} \qquad (0 < r < L),$$

$$\sigma_+(r,\vartheta) = -\frac{2}{\pi\sqrt{r^2-L^2}}\int_0^L \frac{\rho Q(\rho, r, \vartheta)\sqrt{L^2-\rho^2}\, d\rho}{r^2-\rho^2} \qquad (L < r < \infty), \tag{5}$$

$$Q(\rho, r, \vartheta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sigma_-(\rho,\alpha)(r^2-\rho^2)\, d\alpha}{R^2(\rho, r, \vartheta-\alpha)}.$$

When the stress on the crack does not depend on the angular coordinate, the solution becomes simpler: calculation of the Poisson's integral yields the function $Q = \sigma_-(\rho)$, and (5) becomes a solution of the axisymmetric problem [3].

We formulate integral equations for the unknown functions in the case where the distance to the crack side is variable. For this, we first convert Eqs. (3) by dividing the region of integration along the radial coordinate into two parts $(0, L)$ and (L, ∞):

$$\int_L^x \frac{\rho\sigma_n(\rho)}{\sqrt{x^2-\rho^2}}\left(\frac{\rho}{x}\right)^n d\rho = Ax\int_x^\infty \frac{\partial}{\partial\rho}\left[w_n(\rho)\left(\frac{x}{\rho}\right)^n\right]\frac{d\rho}{\sqrt{\rho^2-x^2}} - \int_0^L \frac{\rho\sigma_n(\rho)}{\sqrt{x^2-\rho^2}}\left(\frac{\rho}{x}\right)^n d\rho. \tag{6}$$

Here the parameter L is an arbitrary constant parameter. We note that all functions in (6) do not depend on L, i.e., the introduction of the parameter is an identical transformation. The only requirement is that the right side be considered for $x > L$.

The next manipulation involves inversion, in the interval (L, x), of the Abelian integral operator on the left, after which, for $r > L$, we obtain

$$\sigma_n(r) = A\frac{\Phi_n(r,r)}{\sqrt{r^2-L^2}} - A\int_L^r [\Phi_n(r,x) - \Phi_n(r,r)]\frac{x\, dx}{(r^2-x^2)^{3/2}} - \frac{F_n(r,\vartheta)}{\sqrt{r^2-L^2}},$$

$$\Phi_n(r,x) = \frac{2x}{\pi}\int_x^\infty \frac{\partial}{\partial\rho}\left[w_n(\rho)\left(\frac{x^2}{\rho r}\right)^n\right]\frac{d\rho}{\sqrt{\rho^2-x^2}}, \quad F_n(r,\vartheta) = \frac{2}{\pi}\int_0^L \rho\sigma_n(\rho)\left(\frac{\rho}{r}\right)^n \frac{\sqrt{L^2-\rho^2}\, d\rho}{r^2-\rho^2}.$$

We convert from the Fourier coefficients to the sought functions by multiplying the equality by $\exp(in\vartheta)$. Taking into account that the previously introduced parameter L does not hinder summation of the series, for the functions in the region $r > L$, we obtain

$$\sigma(r,\vartheta) + \frac{1}{\pi^2\sqrt{r^2-L^2}}\int_0^{2\pi}\int_0^L \rho\sigma(\rho,\alpha)\frac{\sqrt{L^2-\rho^2}\, d\rho\, d\alpha}{R^2(\rho, r, \vartheta-\alpha)}$$

$$= A\frac{\Phi(r,r,\vartheta)}{\sqrt{r^2-L^2}} - A\int_L^r [\Phi(r,x,\vartheta) - \Phi(r,r,\vartheta)]\frac{x\, dx}{(r^2-x^2)^{3/2}}, \tag{7}$$

$$\Phi(r,x,\vartheta) = \frac{x}{\pi^2}\int_0^{2\pi}\int_x^{l(\alpha)} \frac{\partial}{\partial\rho}\left[w(\rho,\alpha)\frac{\rho^2 r^2 - x^4}{R^2(x^2, \rho r, \vartheta-\alpha)}\right]\frac{d\rho\, d\alpha}{\sqrt{\rho^2-x^2}}.$$

If $l(\vartheta) = L = \text{const}$, the terms on the right side of Eq. (7) vanish and the second term in the integrand on the left side contains the known stress on the crack σ_-. The stress $\sigma = \sigma_+$ obtained in (7) coincides with solution (5).

Performing similar manipulations for $r < L$, we obtain the following equation for points in the additional region:

$$w(r,\vartheta) - \frac{1}{\pi^2}\sqrt{L^2 - r^2}\int_0^{2\pi}\int_L^{l(\alpha)} \frac{\rho w(\rho,\alpha)\,d\rho\,d\alpha}{R^2(\rho,r,\vartheta-\alpha)\sqrt{\rho^2 - L^2}} = \int_r^L \frac{\Psi(r,x,\vartheta)\,dx}{\sqrt{x^2 - r^2}}, \tag{8}$$

$$\Psi(r,x,\vartheta) = \frac{1}{\pi^2 A}\int_0^{2\pi}\int_0^{x} \rho\sigma(\rho,\alpha)\,\frac{(\rho^2 r^2 - x^4)\,d\rho\,d\alpha}{R^2(x^2,\rho r,\vartheta-\alpha)\sqrt{x^2-\rho^2}}.$$

The displacement is integrated over the region $l(\alpha) > L$.

If the crack contour is a circle, Eq. (8) becomes the first equality in solution (5).

In (7) and (8), we divide the region of integration with respect to the coordinate α into n sectors $(\alpha_k - h_k, \alpha_k + h_k)$. We assume that within each sector, the stress and displacement do not depend on the circumferential coordinate and are equal to their values on the bisectors α_k. The variable step $2h_k$ can be selected so that the curve $\rho = l(\alpha)$ is divided by rays into equal segments. Within each sector, the crack contour is replaced by a circular arc $\rho = l(\alpha_k) = l_k$, where $k = 0, 1, \dots, n$.

Replacing the integrals over the variable α by sums and taking into account that the kernel of the equation is integrable in the kth sector:

$$\int_{\alpha_k - h_k}^{\alpha_k + h_k} \frac{r^2 - \rho^2}{R^2(\rho,r,\vartheta_j - \alpha)}d\alpha = 2\pi\,\mathrm{sign}\,(r-\rho)\delta_{jk} + 2K_{jk}(r,\rho),$$

$$K_{jk}(r,\rho) = \arctan\left(\frac{r-\rho}{r+\rho}\cot\frac{\vartheta_j - \alpha_k - h_k}{2}\right) - \arctan\left(\frac{r-\rho}{r+\rho}\cot\frac{\vartheta_j - \alpha_k + h_k}{2}\right),$$

we obtain the following system of one-dimensional integral equations:

$$\sigma_{j+}(r) + \frac{2}{\pi^2\sqrt{r^2 - l_j^2}}\sum_{k=0}^{n}\int_{l_k}^{l_j} \rho\sigma_{k+}(\rho)K_{jk}(r,\rho)\,\frac{\sqrt{l_j^2 - \rho^2}\,d\rho}{r^2-\rho^2}$$

$$= -\frac{2}{\pi^2\sqrt{r^2 - l_j^2}}\sum_{k=0}^{n}\int_0^{l_k} \rho\sigma_{k-}(\rho)(\pi\delta_{ik} + K_{jk}(r,\rho))\,\frac{\sqrt{l_j^2-\rho^2}\,d\rho}{r^2-\rho^2} + A\,\frac{\Phi_j(r,r)}{\sqrt{r^2 - l_j^2}}$$

$$-A\int_{l_j}^{r}[\Phi_j(r,x) - \Phi_j(r,r)]\,\frac{x\,dx}{(r^2-x^2)^{3/2}} \qquad (r > l_j),$$

$$w_j(r) - \frac{2}{\pi^2}\sqrt{l_j^2 - r^2}\sum_{k=0}^{n}\int_{l_j}^{l_k}\frac{\rho w_k(\rho)K_{jk}(r,\rho)\,d\rho}{(r^2-\rho^2)\sqrt{\rho^2 - l_j^2}} = \int_r^{l_j}\frac{\Psi_j(r,x)\,dx}{\sqrt{x^2-r^2}} \qquad (r < l_j),$$

$$\Phi_j(r,x) = \frac{2x}{\pi^2}\sum_{k=0}^{n}\int_x^{l_k}\frac{\partial}{\partial\rho}[w_k(\rho)K_{jk}(x^2,\rho r)]\,\frac{d\rho}{\sqrt{\rho^2 - x^2}},$$

$$\Psi_j(r,x) = -\frac{2}{\pi^2 A}\sum_{k=0}^{n}\int_0^{x}\rho\sigma_k(\rho)(\pi\delta_{jk} - K_{jk}(x^2,\rho r))\,\frac{d\rho}{\sqrt{x^2-\rho^2}},$$

$$w_j(r) = w(r,\vartheta), \qquad \sigma_j(r) = \sigma(r,\vartheta) \qquad (j = 0, 1, \dots, n)$$

Here δ_{jk} is the Kronecker delta.

In transforming from Eqs. (7) and (8) to the discrete equations, we took into account that the sought solution should not depend on the parameter L, and, hence, within the jth sector, the values of L were set equal to the constant l_j.

The system proposed here has the following advantages.

(1) To determine the stress concentration near the crack, it suffices to "immerse" the crack in a finite region $0 < r < \max_{\vartheta} l(\vartheta) + \text{const}$.

(2) It is possible to convert to the new unknown function $X(r, \vartheta)$:

$$w(r,\vartheta) = \sqrt{l^2(\vartheta) - r^2}\, X(r,\vartheta) \quad [0 \leqslant r \leqslant l(\vartheta)], \qquad \sigma(r,\vartheta) = \frac{Al(\vartheta)X(r,\vartheta)}{\sqrt{r^2 - l^2(\vartheta)}} \quad [l(\vartheta) < r],$$

which is already continuous at points of the smooth contour. On both sides of the contour, this function is obtained from the following unified system of equations:

$$X_j(r) - \frac{2}{\pi^2}\sum_{k=0}^{n}\int_{l_j}^{l_k} \frac{\rho X_k(\rho)K_{jk}(r,\rho)\sqrt{l_k^2 - \rho^2}\,d\rho}{(r^2-\rho^2)\sqrt{\rho^2 - l_j^2}} = \frac{1}{\sqrt{l_j^2 - r^2}}\int_r^{l_j} \frac{\Psi_j(r,x)\,dx}{\sqrt{x^2 - r^2}} \quad (r < l_j),$$

$$l_j X_j(r) + \frac{2}{\pi^2}\sum_{k=0}^{n} l_k \int_{l_k}^{l_j} \frac{\rho X_k(\rho)K_{jk}(r,\rho)\sqrt{l_j^2 - \rho^2}\,d\rho}{(r^2-\rho^2)\sqrt{\rho^2 - l_k^2}} = \Phi_j(r,r) - F_j(r)$$

$$-\sqrt{r^2 - l_j^2}\int_{l_j}^{r} [\Phi_j(r,x) - \Phi_j(r,r)]\frac{x\,dx}{(r^2 - x^2)^{3/2}} \qquad (r > l_j), \tag{9}$$

$$\Psi_j(r,x) = -\frac{2}{\pi^2}\sum_{k=0}^{n}\int_0^x \left[\sigma_{k-}^A(\rho) + \frac{l_k X_k(\rho)}{\sqrt{\rho^2 - l_k^2}}\right](\pi\delta_{jk} - K_{jk}(x^2,\rho r))\frac{\rho\,d\rho}{\sqrt{x^2 - \rho^2}},$$

$$\Phi_j(r,x) = \frac{2x}{\pi^2}\sum_{k=0}^{n}\int_x^{l_k} \frac{\partial}{\partial\rho}\left[X_k(\rho)\sqrt{l_k^2 - \rho^2}K_{jk}(x^2,\rho r)\right]\frac{d\rho}{\sqrt{\rho^2 - x^2}},$$

$$F_j(r) = -\frac{2}{\pi^2}\sum_{k=0}^{n}\int_0^{l_k} \rho\sigma_{k-}^A(\rho)(\pi\delta_{jk} + K_{jk}(r,\rho))\frac{\sqrt{l_j^2 - \rho^2}\,d\rho}{r^2 - \rho^2}/$$

Here $\sigma_{k-}^A = \sigma_{k-}/A$.

In the system of one-dimensional equations (9), the sought quantities on the crack and its extension are "tied up." For example, the interval $r < l_j$ on the right side of the first equation contains unknowns from the interval $\rho > l_k$ for $l_k < l_j$. The right sides of the equations include terms that already take account of the basic features of the solution for varying nonaxisymmetric loading on the circular arcs that approximate the initial contour. Therefore, one can hope for a rapid convergence to the solution in calculations of the present system using the method of successive approximations.

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